

MEASURABLE FUNCTIONS

1. Lebesgue measurable functions

Definition 1.1. Let (X, \mathcal{A}) be a measurable space and $E \in \mathcal{A}$. A function $f : E \rightarrow \overline{\mathbb{R}}$ is said to be *measurable* if for each $\alpha \in \mathbb{R}$, $\{x \in E : f(x) > \alpha\} \in \mathcal{A}$. In particular if $\mathcal{A} = \mathcal{M}$, the class of Lebesgue measurable subsets of \mathbb{R} , then f is called a *Lebesgue measurable function*.

Theorem 1.2. Let E be a measurable subset of X and $f : E \rightarrow \overline{\mathbb{R}}$. Then the following are equivalent:

- (1) f is measurable
- (2) $f^{-1}([\alpha, \infty]) \in \mathcal{A}$, $\forall \alpha \in \mathbb{R}$.
- (3) $f^{-1}([-\infty, \alpha]) \in \mathcal{A}$, $\forall \alpha \in \mathbb{R}$.
- (4) $f^{-1}([-\infty, \alpha]) \in \mathcal{A}$, $\forall \alpha \in \mathbb{R}$.
- (5) $f^{-1}(O) \in \mathcal{A}$ for any open set O of $\overline{\mathbb{R}}$.
- (6) $f^{-1}(F) \in \mathcal{A}$ for any closed set F of $\overline{\mathbb{R}}$.

Proof. We prove (1) \Rightarrow (2). Assume f measurable; we have

$$f^{-1}([\alpha, \infty]) = \bigcap_{n \geq 1} f^{-1}([\alpha - \frac{1}{n}, \infty])$$

is a measurable set as intersection of measurable sets.

We prove (2) \Rightarrow (3). We have $f^{-1}([-\infty, \alpha]) = (f^{-1}([\alpha, \infty]))^c$ is measurable.

We prove (3) \Rightarrow (4). We have $f^{-1}([-\infty, \alpha]) = \bigcap_{n \geq 1} f^{-1}([-\infty, \alpha + \frac{1}{n}])$ is measurable.

We prove (4) \Rightarrow (1). We have $f^{-1}([\alpha, \infty]) = (f^{-1}([-\infty, \alpha]))^c$ is measurable.

(5) \Rightarrow (1) is obvious. We prove (1) \Rightarrow (5). We first note that $f^{-1}([a, b]) = f^{-1}([a, \infty]) \cap f^{-1}([-\infty, b])$ is measurable by (1) and (3). Thus for any open interval I of $\overline{\mathbb{R}}$, $f^{-1}(I)$ is measurable. Let O be an open set of $\overline{\mathbb{R}}$, there is a sequence of open intervals $(I_n)_n$ of $\overline{\mathbb{R}}$ such that $O = \bigcup_n I_n$. Then $f^{-1}(O) = \bigcup_n f^{-1}(I_n)$ is measurable.

(5) \Leftrightarrow (6) by complementation.

Remark 1.3. If f is measurable then, for any $\alpha \in \overline{\mathbb{R}}$, $f^{-1}(\{\alpha\})$ is a measurable set; and for any countable set $A \subset \overline{\mathbb{R}}$, $f^{-1}(A)$ is a measurable set.

Definition 1.4. Let A be a nonempty set and $f : A \rightarrow \overline{\mathbb{R}}$. Then $f^+ := \max(f, 0)$ is called the *positive part* of f and $f^- := \max(-f, 0)$ is called the *negative part* of f .

We have $f^+ \geq 0$, $f^- \geq 0$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Theorem 1.5. Let D be a measurable subset of X , $f, g : D \rightarrow \overline{\mathbb{R}}$ are measurable and $\lambda \in \mathbb{R}$. Then $f + g$ (if it is defined), fg , λf , $\frac{f}{g}$ ($g \neq 0$), $f \vee g$, $f \wedge g$ and $|f|$ are measurable functions.

Remarks 1.6. (1) f is measurable if and only if f^+ and f^- are measurable.

(2) $|f|$ measurable does not imply that f is measurable. Let E be a non measurable set and put $f(x) = 1$ if $x \in E$ and $f(x) = -1$ if $x \in E^c$. We have $|f| = 1$ is

measurable however f is not measurable because $f^{-1}(\{1\}) = E$ is not measurable.

Theorem 1.7. Let $f : D \rightarrow \overline{\mathbb{R}}$ be a measurable function and A be a measurable subset of D . Then $f|_A$, the restriction of f to A , is also measurable.

Corollary 1.8. Let D and E be measurable sets and f a function with domain $D \cup E$. Then f is measurable if and only if $f|_D$ and $f|_E$ are measurable.

Definition 1.9. A property is said to hold almost every where (a.e.) if the set of points where this property fails to hold has measure zero.

Example. $f = g$ a.e. iff $m^*({x : f(x) \neq g(x)}) = 0$.

Theorem 1.10. Let $D \in \mathcal{M}$ and $f : D \rightarrow \overline{\mathbb{R}}$ be a Lebesgue measurable function with $f = g$ a.e. on D . Then g is also Lebesgue measurable.

Proof. $f = g$ on $D \setminus A$ with $m^*(A) = 0$. So A and $D \setminus A \in \mathcal{M}$. Moreover

$$g^{-1}([\alpha, \infty]) = [f^{-1}([\alpha, \infty]) \cap (D \setminus A)] \cup [g^{-1}([\alpha, \infty]) \cap A],$$

where $g^{-1}([\alpha, \infty]) \cap A$ has measure zero and so it is Lebesgue measurable. It follows that $g^{-1}([\alpha, \infty]) \in \mathcal{M}$.

Theorem 1.11. For any $n \geq 1$, let $f_n : D \rightarrow \overline{\mathbb{R}}$ be measurable. Then the functions $\max_{1 \leq n \leq p} f_n$, $\min_{1 \leq n \leq p} f_n$, $\inf_{n \geq 0} f_n$, $\sup_{n \geq 0} f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are all measurable. Moreover, if $\lim_n f_n(x) = f(x)$ exists, then f is measurable.

Remarks 1.12. (1) For arbitrary family $\{f_\alpha\}_{\alpha \in I}$ of measurable functions, $\sup_{\alpha \in I} f_\alpha$ and $\inf_{\alpha \in I} f_\alpha$ may be not measurable. A counter-example: let E be a non measurable set and for each $\alpha \in E$ let $f_\alpha(x) = 1$ if $x = \alpha$ and $f_\alpha(x) = 0$ if $x \neq \alpha$. For any α , f_α is measurable. Put $f = \sup_{\alpha \in E} f_\alpha$. We have $f^{-1}(\{1\}) = E$ is a non measurable set and so f is not measurable.

(2) Suppose $(f_n)_n$ is a sequence of Lebesgue measurable functions such that $f_n \rightarrow f$, a.e. Then f is Lebesgue measurable. This is because $f_n \rightarrow f$ on A^c with $m(A) = 0$ implies $f_n \mathbf{1}_{A^c} \rightarrow f \mathbf{1}_{A^c}$ every where and then by Theorem 1.11, $f \mathbf{1}_{A^c}$ is Lebesgue measurable. Moreover $f \mathbf{1}_A = 0$ a.e. implies $f \mathbf{1}_A$ is Lebesgue measurable, and so $f = f \mathbf{1}_{A^c} + f \mathbf{1}_A$ is Lebesgue measurable.

Theorem 1.13. Let D be in \mathcal{M} and $f : D \rightarrow \mathbb{R}$ be continuous. Then f is Lebesgue measurable.

Proof. For all $\alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty])$ is an open set of D and so $f^{-1}([\alpha, \infty]) = D \cap O$, where O is an open set of \mathbb{R} . Thus $f^{-1}([\alpha, \infty]) \in \mathcal{M}$.

2. Simple functions

Let (X, \mathcal{A}) be a measurable space.

Definition 2.1. Let $A \subseteq X$. The function $\mathbf{1}_A$ defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases}$$

is called *characteristic function* of A .

Theorem 2.2. *The function $\mathbf{1}_A$ is measurable if and only if A is measurable.*

Definition 2.3. Let $A \subset X$. A function $\varphi : A \rightarrow \mathbb{R}$ is said to be a *simple function* if there is a partition $(A_i)_{1 \leq i \leq n}$ of A and real numbers $\alpha_1, \dots, \alpha_n$ such that $\varphi = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$.

In the particular case $A = [a, b]$, $a < b \in \mathbb{R}$, $A_i =]x_{i-1}, x_i[$, where $x_0 = a < x_1 < \dots < x_n = b$, the function φ is called a *step function*.

Theorem 2.4. *Assume that A be a measurable subset of X and $\varphi = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i}$ be a simple function, where $\{A_i\}_{1 \leq i \leq n}$ is a partition of A . Then φ is measurable if and only if A_i is measurable for each $i \in \{1, \dots, n\}$.*

Theorem 2.5. *Let $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function. Then f is the pointwise limit of a sequence $(\varphi_n)_n$ of simple functions. If $f \geq 0$, then $(\varphi_n)_n$ can be taken as monotonic increasing.*

Proof. Let $f \geq 0$. For each integer $n \geq 1$ and $x \in X$ let

$$\varphi_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) \leq \frac{k}{2^n}, k = 1, 2, \dots, n2^n \\ n & \text{if } f(x) \geq n \end{cases}$$

For each integer $n \geq 1$, φ_n is a non-negative simple function and $\varphi_{n+1} \geq \varphi_n$. Moreover if $f(x) < n$, then $0 \leq f(x) - \varphi_n(x) \leq \frac{1}{2^n}$, and if $f(x) = \infty$, then $\varphi_n(x) = n$, $\forall n$. So $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$. If f is only measurable, we consider f^+ and f^- . We have $f^+ = \lim_{n \rightarrow \infty} \varphi_n$ and $f^- = \lim_{n \rightarrow \infty} \psi_n$, where φ_n and ψ_n are simple functions. Thus $f_n := \varphi_n - \psi_n$ is a simple function and $f = \lim_{n \rightarrow \infty} f_n$.

Remark 2.6. We remark that if f is moreover bounded, then $(\varphi_n)_n$ converges uniformly to f .

3. Borel measurable functions

Definition 3.1. Let $(\mathbb{R}, \mathcal{B})$ be the Borel measurable space and $D \subset \mathbb{R}$ be a Borel set. A function $f : D \rightarrow \overline{\mathbb{R}}$ is said *Borel measurable* if for each $\alpha \in \mathbb{R}$, $f^{-1}(] \alpha, \infty]) \in \mathcal{B}$.

Theorem 3.2. *The following statements are equivalent*

- (1) f is Borel measurable
- (2) $f^{-1}(] \alpha, \infty]) \in \mathcal{B}$, $\forall \alpha \in \mathbb{R}$.
- (3) $f^{-1}(] -\infty, \alpha]) \in \mathcal{B}$, $\forall \alpha \in \mathbb{R}$.
- (4) $f^{-1}(] -\infty, \alpha]) \in \mathcal{B}$, $\forall \alpha \in \mathbb{R}$.
- (5) $f^{-1}(V) \in \mathcal{B}$ for every open set V in $\overline{\mathbb{R}}$.
- (6) $f^{-1}(F) \in \mathcal{B}$ for every closed set F in $\overline{\mathbb{R}}$.

Examples

(1) Let $D \in \mathcal{B}$ and $f : D \rightarrow \mathbb{R}$ be continuous. Then f is Borel measurable. This is because, for any open set O of \mathbb{R} , $f^{-1}(O)$ is an open set of D and so $f^{-1}(O) \in \mathcal{B}$.

(2) Let I be an interval and $f : I \rightarrow \mathbb{R}$ be monotone. Then f is Borel measurable. This is because, for any interval J of \mathbb{R} , $f^{-1}(J)$ is equal to a nonempty interval or to

the empty set and so $f^{-1}(J) \in \mathcal{B}$.

Theorem 3.3. *The function f is Lebesgue measurable if and only if for any $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{M}$.*

Proof. Assume that f to be Lebesgue measurable and put

$$\Omega = \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{M}\}.$$

So Ω contains the collection of all open sets. If $A \in \Omega$ then $f^{-1}(A^c) = [f^{-1}(A)]^c \in \mathcal{M}$ and so $A^c \in \Omega$. If $(A_n)_n$ is a sequence in Ω , then $f^{-1}(A_n) \in \mathcal{M}$ and so

$$f^{-1}(\cup_n A_n) = \cup_n f^{-1}(A_n) \in \mathcal{M}.$$

Thus Ω is a σ -algebra containing the collection of open sets, and so $\Omega \supset \mathcal{B}$. The converse is clear.

Theorem 3.4. *The function f is Borel measurable if and only if for all $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{B}$.*

Proof. Assume that f to be Borel measurable and put

$$\Omega = \{A \subset \mathbb{R} : f^{-1}(A) \in \mathcal{B}\}.$$

Ω is a σ -algebra containing the collection of all open sets. Thus $\Omega \supset \mathcal{B}$ and so for all $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{B}$. The converse is clear.

Theorem 3.5. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Lebesgue measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be Borel measurable, then $g \circ f$ is Lebesgue measurable.*

Proof. Let V be an open set of \mathbb{R} ; then $g^{-1}(V) \in \mathcal{B}$, and so by Theorem 3.3

$$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \mathcal{M}.$$

Theorem 3.6. *Let $f : D \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be Borel measurable functions. Then the function $g \circ f : D \rightarrow \overline{\mathbb{R}}$ is Borel measurable.*

Proof. By Theorem 3.4, for all $B \in \mathcal{B}$, $g^{-1}(B) \in \mathcal{B}$ and $f^{-1}(g^{-1}(B)) \in \mathcal{B}$ which means $(g \circ f)^{-1}(B) \in \mathcal{B}$. Thus $g \circ f$ is Borel measurable.

Theorem 3.7. *The Borel algebra \mathcal{B} is a proper subclass of \mathcal{M} , i.e. $\mathcal{B} \subset \mathcal{M}$.*

Proof. Let C be the Cantor set and $f : [0, 1] \rightarrow C$ be the Cantor's function defined by $f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$, where $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$, $a_n = 0$ or 1 , is the binary expansion of x . The function f is bijective and Lebesgue measurable. Let $P \subset [0, 1]$ with $P \notin \mathcal{M}$. Then $A = f(P) \subset C$ with $m^*(A) = 0$ since $m(C) = 0$. Thus $A \in \mathcal{M}$. Assume $A \in \mathcal{B}$, we obtain $P = f^{-1}(A) \in \mathcal{M}$ by Theorem 3.3 which is a contradiction. Thus $A \in \mathcal{M} \setminus \mathcal{B}$.

Remarks 3.8. (1) There is a subset $A \subset C$ with $A \notin \mathcal{B}$.

(2) The Borel measure space $(\mathbb{R}, \mathcal{B}, m)$ is not complete. This is because $A \subset C$ with $C \in \mathcal{B}$, $m(C) = 0$, however $A \notin \mathcal{B}$.

(3) The inverse image of a measurable set by a measurable function is not necessarily measurable. Let $f : [0, 1] \rightarrow C$ be the Cantor's function which is bijective and Lebesgue measurable and $P \subset [0, 1]$ with $P \notin \mathcal{M}$. We have $A = f(P) \subset C$ with $m(A) = 0$, A measurable, however $f^{-1}(A) = P \notin \mathcal{M}$.

(4) The composition of two Lebesgue measurable functions is not necessarily Lebesgue

measurable. Let $f : [0, 1] \rightarrow C$ be the Cantor's function which is bijective and Lebesgue measurable and $A \subset C$ with $A \notin \mathcal{B}$. Since $A \in \mathcal{M}$, $g = \mathbf{1}_A$ is measurable. Moreover $(gof)^{-1}(\{1\}) = f^{-1}(A) = P \notin \mathcal{M}$ and so gof is not Lebesgue measurable.

(5) Every Borel measurable function is Lebesgue measurable function, however the converse is false. For $A \in \mathcal{M} \setminus \mathcal{B}$ the function $\mathbf{1}_A$ is Lebesgue measurable and not Borel measurable.

(6) If f is a Borel measurable function and $f = g$ a.e., then g is not necessarily Borel measurable. Let $A \in \mathcal{M} \setminus \mathcal{B}$, $A \subset C$, the Cantor set. We have $\mathbf{1}_A = 0$ a.e., however $\mathbf{1}_A$ is not Borel measurable.

(7) Suppose $(f_n)_n$ is a sequence of Borel measurable functions such that $f_n \rightarrow f$ a.e. Then f need not be Borel measurable. A counter-example: Let $A \notin \mathcal{B}$, $A \subset C$, where C is the Cantor set. We remark that $f_n(x) = \frac{x}{n} \rightarrow \mathbf{1}_A$ a.e. with f_n Borel measurable and $\mathbf{1}_A$ is not Borel measurable.

4. Convergence and continuity properties of measurable functions

Definition 4.1. Let (X, \mathcal{A}, μ) be a measure space, $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions and $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function. We say that $(f_n)_n$ converges to f , μ -almost uniformly ($f_n \rightarrow f$, μ -a.u.) if for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) < \varepsilon$ and (f_n) converges uniformly to f on A_ε^c .

Theorem 4.2. Let (X, \mathcal{A}, μ) be a measure space, $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions and $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function. If $(f_n)_n$ converges to f , μ -almost uniformly, then $(f_n)_n$ converges to f , μ -almost everywhere.

Proof. For $\varepsilon = \frac{1}{n}$, there exists $A_n \in \mathcal{A}$ with $\mu(A_n) < \frac{1}{n}$ and (f_n) converges uniformly to f on A_n^c . The set $A = \bigcap_n A_n \in \mathcal{A}$ with $\mu(A) = 0$ and (f_n) converges pointwisely to f on A^c .

Theorem 4.3 (Egorov's theorem). Let (X, \mathcal{A}, μ) be a measure space, with $\mu(X) < \infty$ and $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions. Assume that for each n , f_n is finite μ -a.e. and (f_n) converges to f , μ -almost everywhere with f is finite μ -a.e. Then (f_n) converges to f , μ -almost uniformly.

Proof. We first show that for all $\varepsilon > 0$, $\delta > 0$ there exists $A \in \mathcal{A}$ and $n_0 \in \mathbb{N}$ with $\mu(A) < \delta$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall x \in A^c$ and $\forall n \geq n_0$. Since f_n is measurable and $f_n \rightarrow f$, μ -a.e., f is measurable. Let

$$E = \{x \in X : f(x) \in \mathbb{R}, f_n(x) \rightarrow f(x)\}$$

which is measurable. For $\varepsilon > 0$ and $k \in \mathbb{N}^*$, let

$$E_k = \{x \in E : |f_n(x) - f(x)| < \varepsilon, \forall n \geq k\}.$$

Then $(E_k)_k$ is an increasing sequence in \mathcal{A} with $E = \bigcup_{k=1}^{\infty} E_k$ and $\mu(E^c) = 0$. Since $\mu(E) < \infty$, for any $\delta > 0$, there exists n_0 such that $\mu(E \setminus E_{n_0}) < \delta$. Put $A = (E \setminus E_{n_0}) \cup E^c$. We have $A \in \mathcal{A}$, $\mu(A) < \delta$ and $x \in A^c = E_{n_0} \Rightarrow |f_n(x) - f(x)| < \varepsilon$, $\forall n \geq n_0$.

For $\varepsilon = \frac{1}{p}$, $p \in \mathbb{N}^*$ and $\delta > 0$, there exists $A_p \in \mathcal{A}$ with $\mu(A_p) < \frac{\delta}{2^p}$ and n_p such that

$|f_n(x) - f(x)| < \frac{1}{p}, \forall n \geq n_p, \forall x \in A_p^c$. Let $A = \cup_{p \geq 1} A_p$. We have $A \in \mathcal{A}, \mu(A) < \delta$ and $\lim_{n \rightarrow \infty} \sup_{x \in A^c} |f_n(x) - f(x)| = 0$.

Definition 4.4. Let (X, \mathcal{A}, μ) be a measure space, $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions and $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function. We say that $(f_n)_n$ converges uniformly μ -almost everywhere to f ($f_n \rightarrow f$ u. μ -a.e.) if there exists $A \in \mathcal{A}$ with $\mu(A) = 0$ and (f_n) converges uniformly to f on A^c .

Remark 4.5. Clearly, the uniform convergence \Rightarrow the uniform convergence μ -almost everywhere \Rightarrow the μ -almost uniform convergence. The sequence $f_n(x) = x^n, x \in [0, 1]$ converges almost uniformly to 0 but it does not converge uniformly almost everywhere to 0.

Theorem 4.6 (Lusin's theorem). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lebesgue measurable function. For any $\varepsilon > 0$, there exists a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ and a closed set $F \subset \mathbb{R}$ with $m(F^c) < \varepsilon$ such that $f = g$ on F .

To prove the theorem we first need to recall the Urysohn's lemma and the Tiestze extension theorem.

Theorem (Urysohn's lemma). Let X be a normal space; let A and B be disjoint closed subsets of X . Then there exists a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$.

Theorem (Tiestze extension theorem). Let X be a normal space and A be a closed subspace of X . Then any continuous map of A into \mathbb{R} may be extended to a continuous map of all of X into \mathbb{R} .

Proof of Theorem 4.6. Case 1: $f = \mathbf{1}_A, A \in \mathcal{M}$. For any $\varepsilon > 0$, there is an open set V and a closed set E with $E \subset A \subset V, m(V \setminus A) < \frac{\varepsilon}{2}$ and $m(A \setminus E) < \frac{\varepsilon}{2}$. By Urysohn's lemma, there exists a continuous function $g : \mathbb{R} \rightarrow [0, 1]$ such that $g = 1$ on E and $g = 0$ on V^c . So $g = f$ on $F := E \cup V^c$ a closed set, and $m(F^c) = m(V \setminus E) < \varepsilon$.

Case 2: The general case: $f = \sum_{n=1}^p \alpha_n \mathbf{1}_{A_n}, A_n \in \mathcal{M}$. By Case 1, for any $\varepsilon > 0$, there exists $g_n \in C(\mathbb{R})$ and a closed set F_n such that $g_n = \mathbf{1}_{A_n}$ on F_n and $m(F_n^c) < \frac{\varepsilon}{p}$. Let $g = \sum_{n=1}^p \alpha_n g_n$. So $g \in C(\mathbb{R})$ with $f = g$ on $F = \cap_{n=1}^p F_n$, a closed set, and $m(F^c) < \sum_{n=1}^p \frac{\varepsilon}{p} = \varepsilon$.

Case 3: Assume that f is a bounded Lebesgue measurable function. There is a sequence $(f_n)_n$ of Lebesgue measurable simple functions which converges to f uniformly on \mathbb{R} . Let $\varepsilon > 0$. By Case 2, for each $n \geq 1$ there exists a function $g_n \in C(\mathbb{R})$ and a closed set F_n such that $f_n = g_n$ on F_n and $m(F_n^c) < \frac{\varepsilon}{2^n}$. The set $F = \cap_n F_n$ is closed, $(g_n)_n$ converges uniformly to f on F and $m(F^c) < \varepsilon$. Hence the function $f|_F : F \rightarrow \mathbb{R}$ is continuous, and so by the Tiestze extension theorem there exists $g \in C(\mathbb{R})$ such that $g = f$ on F .

Case 4: The general case. Let $f_n = f$ if $|f| \leq n$ and $f_n = n$ if $|f| > n$. By Case 3, for each n , there is $g_n \in C(\mathbb{R})$ and a closed set F_n such that $f_n = g_n$ on F_n and $m(F_n^c) < \frac{\varepsilon}{2^{n+1}}$. The set $F = \cap_n F_n$ is closed, $m(F^c) < \varepsilon/2$ and $(g_n)_n$ converges to f on F . By Egorov's theorem for each $p \geq 1$ there is an open set $O_p \subset F \cap [-p, p]$ such that

$m(O_p) < \frac{\varepsilon}{2^{p+1}}$ and (g_n) converges uniformly to f on $F \cap [-p, p] \setminus O_p$. Put $O = \cup_p O_p$, an open set. Thus f is continuous on $F \cap [-p, p] \setminus O$ for each p and so it is continuous on $\cup_p F \cap [-p, p] \setminus O = F \setminus O = E$ closed set with $m(E^c) \leq m(F^c) + m(O) < \varepsilon$. By the Tiestze extension theorem there exists $g \in C(\mathbb{R})$ such that $g = f$ on E .

Corollary 4.7. *Let I be an interval and $f : I \rightarrow \mathbb{R}$ Lebesgue measurable. For any $\varepsilon > 0$, there exists an open set $V \subset I$ with $m(V) < \varepsilon$ and a continuous function $g : I \rightarrow \mathbb{R}$ such that $f = g$ on V^c .*

Proof. Apply Theorem 4.6 to the function $f\mathbf{1}_I$ defined on \mathbb{R} .

Corollary 4.8. *Let I be an interval and $f : I \rightarrow \mathbb{R}$ Lebesgue measurable. Then there exists a sequence of continuous functions on $g_n : I \rightarrow \mathbb{R}$ such that $f(x) = \lim_n g_n(x)$, a.e. x .*

Proof. By Corollary 4.7, for each $n \geq 1$, there exists an open set $V_n \subset I$ with $m(V_n) < 1/2^n$ and a continuous function $g_n : I \rightarrow \mathbb{R}$ such that $f = g_n$ on V_n^c . Since $\sum_n m(V_n) < \infty$, by Borel-Cantelli lemma $m(\limsup_n V_n) = 0$. So by letting $A = \limsup_n V_n$, we have $m(A) = 0$ and $\lim_n g_n = f$ on A^c .